Session 10 : solutions

Problem : show that terms of the kind

1) $\left(\nabla^{\frac{n}{2}}\phi\right)^2$ $\quad$ n even

and terms of the kind

2) ϕ^n $\quad$ n even

are irrelevant if we choose

$$\tilde{\phi}\left(\frac{\vec{k}}{b}\right) = \tilde{\phi}(\vec{k})\, b^{\lambda} \qquad \text{with} \qquad \lambda = \frac{d+2}{2}$$

1) The corresponding term in the Hamiltonian is

$$\int d\vec{x}\, \left(\nabla^{n/2}\phi\right)^2$$

Apply Fourier transform :

$$\phi(\vec{x}) = \frac{1}{(2\pi)^d}\int e^{i\vec{k}\vec{x}}\,\tilde{\phi}(\vec{k})\, d\vec{k}$$

then

$$\int d\vec{x} \left[\nabla^{n/2}\int \frac{d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}} \tilde{\phi}(\vec{k}) \cdot \nabla^{n/2}\int \frac{d\vec{k}'}{(2\pi)^d} e^{i\vec{k}'\vec{x}}\tilde{\phi}(\vec{k}')\right] =$$

$$= \int d\vec{x}\, (i)^n \int \frac{d\vec{k}}{(2\pi)^d}\int \frac{d\vec{k}'}{(2\pi)^d} \vec{k}^{n/2} \tilde{\phi}(\vec{k})\, \vec{k}'^{n/2} \tilde{\phi}(\vec{k}')\, e^{i(\vec{k}+\vec{k}')\vec{x}} =$$

$$= (i)^n \int \frac{d\vec{k}}{(2\pi)^d} \frac{d\vec{k}'}{(2\pi)^d}\, \vec{k}^{n/2}\, \vec{k}'^{n/2}\, \tilde{\phi}(\vec{k})\tilde{\phi}(\vec{k}') \int d\vec{x}\, e^{i(\vec{k}+\vec{k}')\vec{x}} =$$

$$= (i)^n \frac{1}{(2\pi)^d} \int d\vec{k}\, d\vec{k}'\, \delta(\vec{k}+\vec{k}')\, \vec{k}^{n/2}\, \vec{k}'^{n/2}\, \tilde{\phi}(\vec{k})\tilde{\phi}(\vec{k}') =$$

$$= (-i)^n \frac{1}{(2\pi)^d} \int d\vec{k}\; k^n\, \tilde{\phi}(\vec{k})\tilde{\phi}(-\vec{k}) =$$

$$= (-i)^n \frac{1}{(2\pi)^d} \int d\vec{k}\; k^n\, \tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})$$

$$\Big\downarrow \text{ after shell integration :}$$

$$(-i)^n \frac{1}{(2\pi)^d} \int_0^{\Lambda/b} d\vec{k}\; k^n\, \tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k}) =$$

$$= (-i)^n \frac{1}{(2\pi)^d}\, b^{-d-n+2\lambda} \int_0^{\Lambda} d\vec{k}\; k^n\, \tilde{\phi}(\vec{k})\tilde{\phi}^*(\vec{k})$$

choose $\lambda = \frac{d+2}{2}$

$$\Rightarrow \quad b^{-d-n+d+2} = b^{2-n}$$

Then if $n = 2 \longrightarrow b^0$

But if $n > 2 \longrightarrow b^{2-n} = \frac{1}{b^{n-2}}$

and since $b > 1$, these terms go to 0 after repeated coarse grainings

2)

$$\int d\vec{x}\ \phi^n(\vec{x}) \underset{\substack{\uparrow \\ \text{Fourier}}}{=} \int d\vec{x}\ \frac{1}{(2\pi)^{nd}} \left[\int d\vec{k}\ e^{i\vec{k}\vec{x}}\, \tilde{\phi}(\vec{k}) \right]^n =$$

$$= \int \prod_{i=1}^{n} d\vec{k}_i \frac{1}{(2\pi)^{nd}} \prod_{i=1}^{n} \tilde{\phi}(\vec{k}_i) \int d\vec{x}\ e^{i\vec{k}\vec{x}} =$$

$$= \frac{1}{(2\pi)^{n(d-1)}} \int_0^{\Lambda} \prod_{i=1}^{n} d\vec{k}_i \prod_{i=1}^{n} \tilde{\phi}(\vec{k}_i)\ \underbrace{\delta\left(\sum_i \vec{k}_i\right)}_{\substack{\text{constraint : reduces} \\ \text{by 1 the number} \\ \text{of degrees of freedom}}}$$

After shell integration, the term becomes

$$\int_0^{\Lambda/b} \prod_{i=1}^{n} d\vec{k}_i \, \delta\left(\sum_i \vec{k}_i\right) \prod_{i=1}^{n} \tilde{\phi}(\vec{k}_i) =$$

$$= b^{-(n-1)d + n\lambda} \int_0^{\Lambda} \prod_{i=1}^{n} d\vec{k}_i \, \delta\left(\sum_i \vec{k}_i\right) \prod_{i=1}^{n} \tilde{\phi}(\vec{k}_i)$$

Now, if we choose $\lambda = \frac{d+2}{2}$ we have

$$b^{-nd + d + n\frac{d}{2} + n} = b^{n\left(1 - \frac{d}{2}\right) + d}$$

When is $n\left(1 - \frac{d}{2}\right) + d < 0$ so that the term is irrelevant upon repeated rescaling?

We have $d > \frac{2n}{n-2}$

n	d
2	2
4	4
6	3
8	$8/3 < 3$
$\vdots$	
∞	2

If $d \simeq 4$, only the term of order $n=4$ is relevant (see $4-\varepsilon$ expansion) but all the others are irrelevant